# Why integrate?

## 1 Introduction

Teaching physics my colleagues and I often find that some of you find it difficult to see when you need to use integration. This session will explain the need for integration using some basic ideas and examples. There will be a second session on integration in semester 2 looking at some more involved issues. Here we concentrate on when to integrate and how to set up integrals in a physics context.

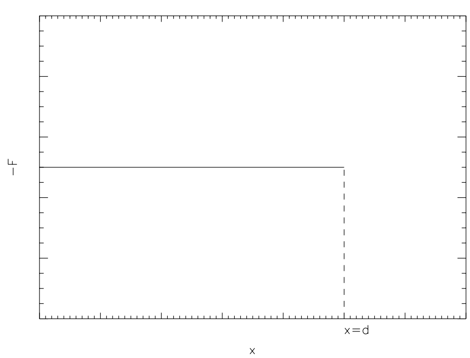
## 2 Integrating constant functions

Consider a car driving along the positive $x$-axis maintaining a constant velocity $v$. A force due to friction of the tyres on the road acts on the car and can be expressed as $F = -kv$, where $k$ is a constant.

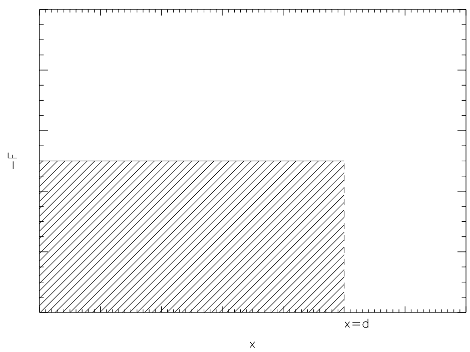
Why is there a minus sign in this definition? (The minus sign indicates that the force is acting in a direction opposite the velocity of the car, i.e. it slows the car down.)

To maintain the constant velocity $v$ the car has to do work. What is the total work done by the car travelling from $x = 0$ to $x = d$? (The work done is equal to $W = -Fd = kvd$. Note that this is a *special case*! Firstly, normally there is an integral involved in calculating $W$ from $F$ along $d$, see below. Secondly, the force is a *vector* and in general the product of $F$ and the distance $d$ is a *dot-product* between two vectors!)

In this case the force is constant along the way and we can easily make a plot of this:



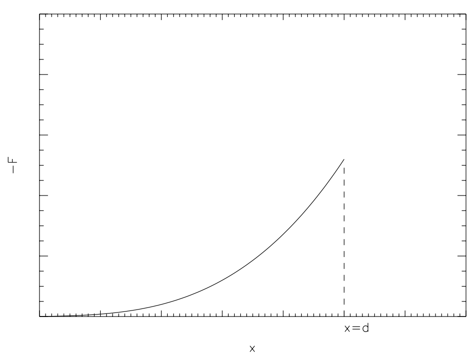
The work done by the car is simply the area under the solid line, i.e. the hatched area in this plot:



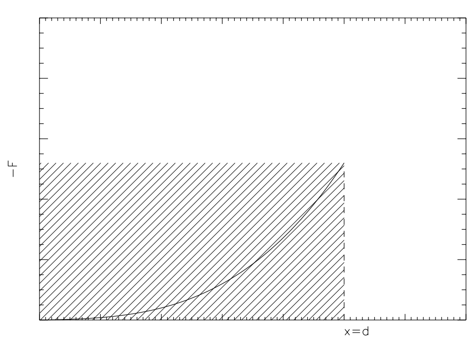

# 3 Integrating a non-constant function

The force needed to extend a spring by a length $x$ is usually given by $F = -kx$, where $k$ is now the spring constant. If we extend a spring a very long way, then the force becomes non-linear, e.g. $F = -kx - ax^3$.

How much work do we need to do to extend this non-linear spring from $x = 0$ to $x = d$? Let's first plot the force:

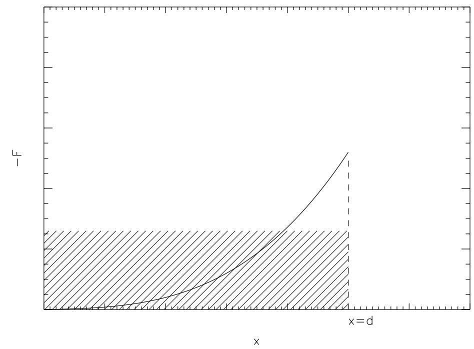



If we now simply multiply the value of the force at the largest extension, $x = d$, with the magnitude of the force at this point, $F(d) = -kd - ad^3$, we get the answer wrong as shown in this plot:



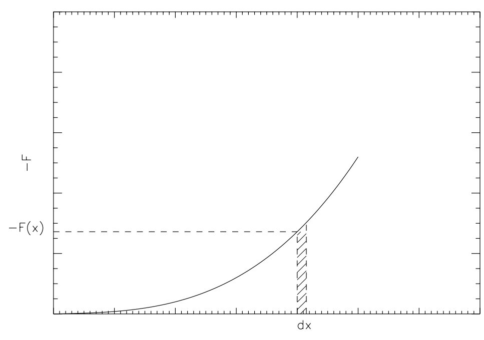

Why does this not work? (The magnitude of the force *changes* along the way.)

A lot of people realise this and so try to multiply the extension of the spring with an intermediate magnitude of the force, say half-way between $x = 0$ and $x = d$. The thinking behind this is something like: ‘The force changes its magnitude along the way. So I can’t multiply the extension with the magnitude of the force at the very beginning nor with the magnitude of the force at the very end. So, something in the middle should work.’ Well, it doesn’t as shown in this plot:




You have to be *very* lucky to just hit the correct answer this way!

Where is the problem? The problem is that the force above is not constant along the entire way. However, we can split the distance $d$ into lots of tiny parts of length $\mathrm{d}x$. Mathematically we can make these *differentials* $\mathrm{d}x$ as small as we like. Let’s have a look at this in a plot:

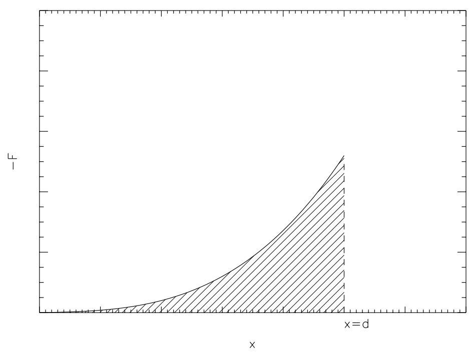



In this plot you can probably still see a small difference between the magnitude of the force at $x$ and the one at $x + \mathrm{d}x$, but it’s not very big. If I make $\mathrm{d}x$ *much* smaller than the one plotted here, i.e. $\mathrm{d}x \to 0$, then the difference of the magnitude of the force at $x$, $F(x)$, and the magnitude of the force at $x + \mathrm{d}x$, $F(x + \mathrm{d}x)$, also becomes very small, i.e. $F(x) - F(x\mathrm{d}x) \to 0$. So, over the very small distance $\mathrm{d}x$ the magnitude of the force is constant. This means I can work out the very small amount of work done by extending the spring from $x$ to $x + \mathrm{d}x$ as $\mathrm{d}W = -F(x)\,\mathrm{d}x$.

If I now want to work out the total amount of work done from $x = 0$ to $x = d$, I need to sum up all the tiny contributions $\mathrm{d}W$ along the way. This is then simply an integration:

$$W = -\int_0^d F\,\mathrm{d}x.$$

The solution of the integral is fairly straightforward. Graphically what we now get is:




The trick with setting up an integral is to spot the correct variable to integrate over. Let's look at some more examples.

# 4 Surface integrals

I'm an astronomer, so let's look at an astronomy example. Many galaxies like the Milky Way are essentially a circular disc. We can measure the density of matter (stars, gas, ...) by a variety of observations. To first approximation we often find that the surface density, i.e. mass per unit area, in the disc is a function of radius, $r$. A simple example would be

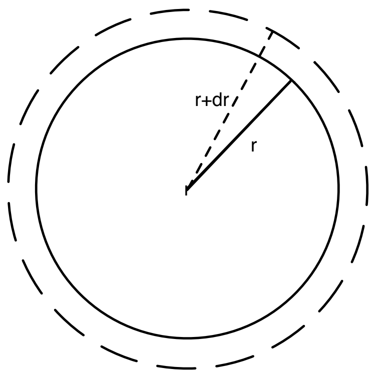
$$n = n_0 \left(\frac{r}{r_0}\right)^{-1},$$

where clearly $n_0$ is the surface density at the scale radius $r_0$.

We now want to work out the mass of the matter in this disc between $r = 0$ and $r = r_{\max}$. Given what we said before, it may be a good idea to define a very small area of the disc, $\mathrm{d}A$, over which $n$ does not vary so that we can work out the mass of this very small area as $\mathrm{d}m = n\,\mathrm{d}A$. We can then sum up all the contributions $\mathrm{d}m$ from all the very small $\mathrm{d}A$ that make up the disc. Clearly this leads again to an integral.

The question now is, how best to divide up the disc into lots of $\mathrm{d}A$? We *could* use a Cartesian co-ordinate system with variables $x$ and $y$. This would allow us to define $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$, but as you can see, that means we need to integrate over *two* variables, $x$ and $y$. Also, we need to work out $r$ as a function of $x$ and $y$ to use the correct value of $n$ for each combination of $x$ and $y$. While this is perfectly possible, it makes life difficult.

From the expression for $n$ we may note that $n$ is constant for constant $r$. This defines a circular ring on the disc where $n$ is constant. If we now move out by a very small distance $\mathrm{d}r$, then the surface density will be almost constant over the area of the ring between $r$ and $r + \mathrm{d}r$. The plot illustrates this:




What is the area of the ring between $r$ and $r+\mathrm{d}r$? The simplest idea is to take the circumference of the inner boundary of the ring, $2\pi r$, and multiply this by the thickness of the ring, $\mathrm{d}r$ giving $2\pi r\,\mathrm{d}r$. Why does this work? The area of the disc inside $r+\mathrm{d}r$ is $\pi(r+\mathrm{d}r)^2$. The area of the disc inside $r$ is $\pi r^2$. The area of the ring must then be $\pi(r+\mathrm{d}r)^2-\pi r^2 = 2\pi r\,\mathrm{d}r+\pi(\mathrm{d}r)^2$. The differential $\mathrm{d}r$ is very small, certainly compared to $r$. Therefore we can neglect the term $\pi(\mathrm{d}r)^2$ compared to $2\pi r\,\mathrm{d}r$ and so the area of the ring is $2\pi r\,\mathrm{d}r$ as before.

The very small contribution to the total mass by the ring is then $\mathrm{d}m = n\,\mathrm{d}A = n\,2\pi r\,\mathrm{d}r$. The total mass must be

$$m = \int n\,\mathrm{d}A = \int_0^{r_{\max}} n\,2\pi r\,\mathrm{d}r = 2\pi n_0 r_0 \int_0^{r_{\max}} \mathrm{d}r = 2\pi n_0 r_0 r_{\max}.$$

Integrating over $r$ makes life *much* simpler!

## 5 Volume integrals

The same principle often works in 3D as well. Let's work out the mass of a spherical star extending to a radius $R_{\max}$ with a density distribution (this time this is mass per unit volume) of

$$\rho = \rho_0 \left(\frac{R}{R_0}\right)^{-1}.$$

This time we need to define very small volume elements $\mathrm{d}V$ over which $\rho$ does not vary. Again we could do this in Cartesian co-ordinates, $x$, $y$ and $z$, and find the answer, but this time this would require integrating over three variables!

The much simpler way to do this is by realising that the density is constant as a function of $R$, which defines a sphere with radius $R$ on the surface of which $\rho$ does not vary. We can then define a spherical shell by extending $R$ by a very small amount $\mathrm{d}R$. What is the volume of this very thin spherical shell?

The simplest way of doing this is to multiply the surface area of the inner boundary of the shell, $4\pi R^2$, with the thickness of the shell, $\mathrm{d}R$. Thus $\mathrm{d}V = 4\pi r^2\,\mathrm{d}r$. Why is this correct? The volume of the sphere inside $R+\mathrm{d}R$ is $4/3\pi(R+\mathrm{d}R)^3$. The volume inside $R$ is $4/3\pi R^3$. Therefore the volume of the spherical shell between $R$ and $R+\mathrm{d}R$ must be

$$\mathrm{d}V = \frac{4}{3}\pi(R+\mathrm{d}R)^3 - \frac{4}{3}\pi R^3 = \frac{4}{3}\pi\left[3R^2\,\mathrm{d}R + 3R(\mathrm{d}R)^2 + (\mathrm{d}R)^3\right].$$

Again we neglect terms containing $(\mathrm{d}R)^2$ or even $(\mathrm{d}R)^3$ and so again $\mathrm{d}V = 4\pi R^2\,\mathrm{d}R$.

The contribution to the total mass by the spherical shell is now $\mathrm{d}m = \rho\,\mathrm{d}V$. The total mass is given by

$$m = \int \rho\,\mathrm{d}V = \int_0^{R_{\max}} \rho\, 4\pi R^2\,\mathrm{d}R = 4\pi\rho_0 R_0 \int_0^{R_{\max}} R\,\mathrm{d}R = 2\pi\rho_0 R_0 R_{\max}^2.$$

Again this is *much* simpler than trying to do this in Cartesian co-ordinates.

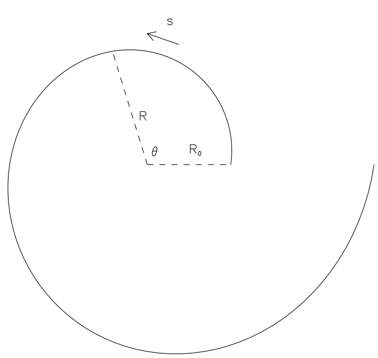
## 6 Line integrals

As mentioned before, the trick with integration is often to spot the best variable to integrate over. Let's look at the next astronomy example.

Galaxies like the Milky Way are not perfect discs, but they contain spiral patterns. Assume that all the mass in the Milky Way is concentrated along a very narrow spiral described by

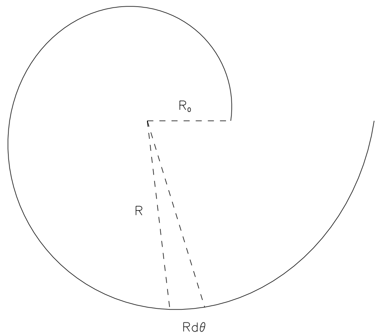
$$R = R_0 \exp(k\theta),$$

where $k$ is a constant. This looks like this:




If we follow the spiral arm for one turn around the centre of the Milky Way, how long is the path we have travelled? The variable $s$ measures the length of the path along the spiral. So for $\theta = 0$ we have $R = R_0$ and $s = 0$. The above question then boils down to: 'What is the value of $s$ at $\theta = 2\pi$?'

As before, we can in principle work this out using Cartesian co-ordinates. But this one would be *very* tricky! Consider travelling a very short way d$s$ along the spiral. Over this very short distance the value of $R$ does not change. Can we express d$s$ in terms of $R$? Consider this plot:




The differential d$s$ can be written as $R\,\mathrm{d}\theta$. But this is all we need since we have a relation between $R$ and $\theta$ along the spiral. The length of our path is the sum of all the very short distances d$s$,

$$s = \int \mathrm{d}s = \int_0^{2\pi} R\,\mathrm{d}\theta = \int_0^{2\pi} R_0 \exp(k\theta)\,\mathrm{d}\theta = \frac{R_0}{k}[\exp(2\pi k) - 1].$$

# 7 Final remarks

The tricky bit with integrals is to set them up. From the examples above we see that usually we can think of integrals in terms of summing very small contributions to give a grand total. The contributions are very small to ensure that a function stays constant within the small contribution while it may vary considerably over the entire range of the sum/integral. It is often difficult to spot the best variable to integrate over. The best way to spot the best choice is to draw diagrams and think about how the various functions vary with the possible choice of variables. In most cases there is some kind of symmetry in the problem that makes the function constant along lines or on surfaces. If we can find such lines or surfaces, then we can usually find a way of simplifying the integral.

Written by Christian Kaiser, 2007.